POLS571 - Longitudinal Data Analysis
October 9, 2001

Error-Correction Models

1 Introduction

So far, we've seen a couple different approaches for modeling (lagged) relationships in nonstationary time-series, both of which have their problems. If we don't difference the series,

$$Y_t = \alpha + \beta X_{t-1} + u_t \tag{1}$$

we run the risk of autocorrelated errors and spurious regressions. If we *do* difference the series, however, we have:

$$\Delta Y_t = \alpha + \beta \Delta X_{t-1} + u_t \tag{2}$$

in which case the effects of X on Y are purely of a short-term nature. But, what if we believe that X and Y are both connected in the short-term, but also more generally related over the long haul? Neither model fits the bill.

2 ECMs Explained

The idea behind *error-correction models* is that there is a long-term equilibrium relationship between X and Y. Short-term "shocks" disturb this relationship, after which the two variables return to equilibrium. Durr (1991) writes a very simple error-correction model as:

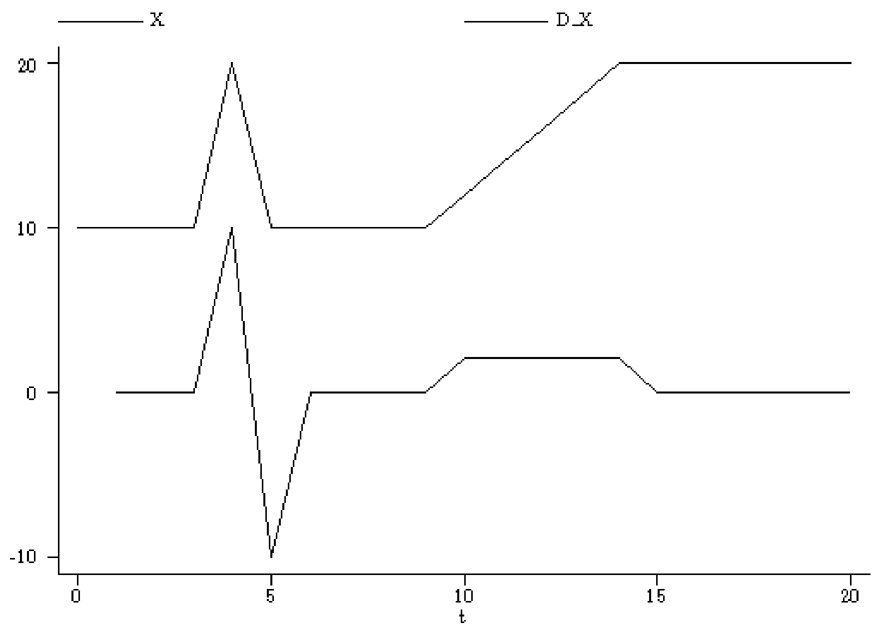
$$\Delta Y_t = \beta \Delta X_{t-1} + \rho(Y_{t-1} - \alpha - \gamma X_{t-1}) \tag{3}$$

In this model there are three important sets of terms:

- $\hat{\beta}$ captures the short-term relationship between X and Y.
- $\hat{\alpha}$ and $\hat{\gamma}$ capture the long-term relationship between X and Y (the "attractor" – the equilibrium distance between X and Y).

- $\hat{\rho}$ gives the rate at which the model "reequilibrates", that is, the speed with which it returns to its equilibrium level. Formally, $\hat{\rho}$ tells us the proportion of the disequilibrium which is corrected with each passing period.

Figure 1: Example: X and ΔX



Consider the following example data. Here, X shows a sharp "spike" at $t = 4$; then, at $t = 10$, it increases slowly from 10 to 20, where it stays permanently. Here are two different ECMs:

$$\Delta Y_t = 1.0\Delta X_{t-1} + 0.8(Y_{t-1} - 5.0 - 1.0X_{t-1}) \tag{4}$$

$$\Delta Y_t = 0.25\Delta X_{t-1} + 0.2(Y_{t-1} - 10.0 - 2.0X_{t-1}) \tag{5}$$

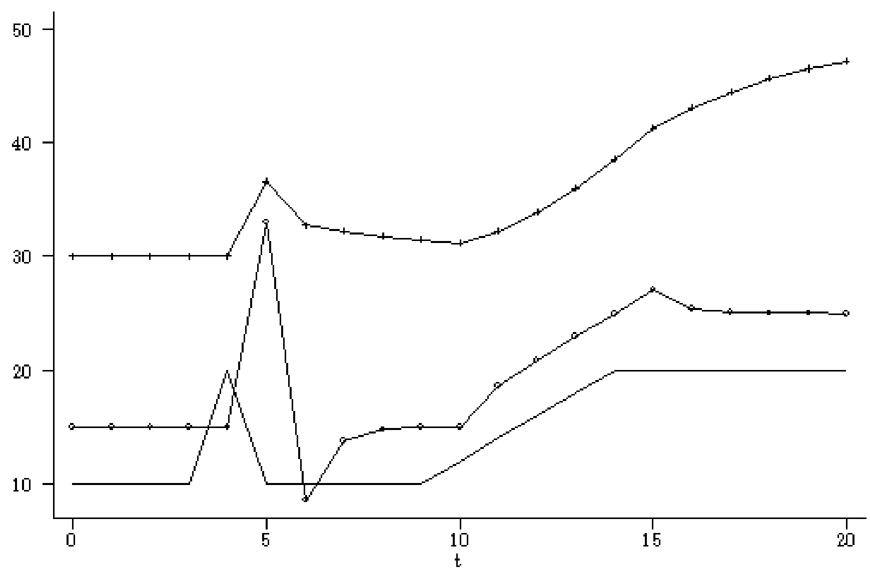
Note a couple things about these models:

- (4) has a larger degree of short-term dependence of X on Y.

- The "equilibrium distance" in (4) is $Y_t = 5 + X_t$; that in (5) is $Y_t = 10 + 2X_t$.
- Model (4) reequilibrates at a much faster rate than does (5).

From the models here, and the X variable discussed above, I simulated Y for each of the two series; these are displayed in Figure 2.

Figure 2: Example: Simulated Ys from ECM Models



Note that:

- The larger degree of short-term dependence in (4) can be seen in the greater extent to which it reacts to immediate changes in X.
- For, e.g., $X = 10$, the equilibrium levels are $Y = 15$ and 30, for the two series, respectively; for $X = 20$, the equilibria are $Y = 25$ and 50.
- Series (4) shows the faster return to equilibrium than does (5).
- Series (4) tends to "overreact" a bit; this is due to the relatively large value of $\hat{\rho}$.

3 Estimation and Testing

Last time, we noted that if both X_t and Y_t are cointegrated, we can write:

$$\begin{aligned} X_t &= W_t + u_{Xt} \\ Y_t &= AW_t + u_{Yt}, \\ W_t &\sim I(1), \\ u_{Xt}, u_{Yt} &\sim I(0) \end{aligned}$$

In these series, the common variance of X and Y are due to a common I(1) component W_t. Moreover, we noted that $Z_t = X_t - AY_t$ was the "attractor" that summarized this equilibrium relationship. That is, the long-term dependence of Y on X (due to W) is captured by the variable Z_t. If, in addition, there is short-term dependence between X and Y, we might write this as:

$$\Delta Y_t = \beta_0 + \beta_1 \Delta X_{t-1} + \beta_2 \Delta X_{t-2} + ... + \beta_k \Delta X_{t-k} + \rho Z_{t-1} + u_t \quad (6)$$

where the terms for the ΔXs capture the nature of the short-term dependence between X and Y. This equation can be rewritten in turn as:

$$\Delta Y_t = \beta_0 + \beta_1 \Delta X_{t-1} + \beta_2 \Delta X_{t-2} + ... + \beta_k \Delta X_{t-k} + \rho(X_t - \alpha Y_t) + u_t \quad (7)$$

which is the ECM model we've been talking about all day. This formulation of the model suggests two possible alternatives for estimating ECMS...

3.1 The Engle-Granger Two-step Method

Nor surprisingly, Engle and Granger's approach is one based in the close connection they draw between ECMs and cointegration. The basics of the approach boil down to three steps:

1. Estimate the cointegrating regression $Y_t = \alpha + \gamma X_t + e_t$,
2. From these estimates, generate $\hat{Z}_t = Y_t - \hat{\alpha} - \hat{\gamma} X_t$,
3. Include $\hat{Z}_{t-1}$ for Z_{t-1} in model (6), above.

Example: House and Senate Composition Data

Going back to our Exercise 1 data, we can estimate the ECM of House on Senate democratic membership, as follows. First, estimate the cointegrating regression:

```
. reg dhpct dspct
```

which yields:

$$\text{House Pct.}_t = 16.46 + 0.728(\text{Senate Pct.})_t + e_t$$

In equilibrium, we estimate that the Senate is more Democratic than the House. From this equation, we can get the residuals $\hat{Z}_t$, the lagged values of which we then plug into the ECM along with the lags of ΔX:

```
. predict Zt, resid

. gen lagZ=Zt[_n-1]
(1 missing value generated)

. dif 1 dhpct

. dif 1 dspct

. lag 1 D_dspct

. reg D_dhpct LD_dspct lagZ
```

which yields:

$$\Delta HousePct._t = \underset{(1.01)}{0.12} - \underset{(0.12)}{0.02}(\Delta SenatePct.)_{t-1} - \underset{(0.14)}{0.60}(Z_{t-1}) + u_t$$

This means that the relationship between the two returns to its equilibrium levels at a rate of about 60 percent of the disequilibrium per Congress.

3.2 One-Step Methods

Note that, with one lag of ΔX, if we substitute $Z_t = X_t - AY_t$ into (6), we get:

$$\begin{aligned} \Delta Y_t &= \beta_0 + \beta_1 \Delta X_{t-1} + \rho(Y_{t-1} - \alpha - \gamma X_{t-1}) + u_t \\ &= (\beta_0 - \rho\alpha) + \beta_1 \Delta X_{t-1} + \rho Y_{t-1} - \rho\gamma X_{t-1} + u_t \end{aligned} \quad (8)$$

This suggests that we can estimate an ECM as a single equation, where changes in Y_t are a function of:

- Lagged changes in X,
- the one-period lagged level of X, and
- the one-period lagged level of Y.

We can do this easily in Stata:

```
.  lag 1 dhpct

.  lag 1 dspct

.  reg D_dhpct LD_dspct L_dhpct L_dspct
```

which yields:

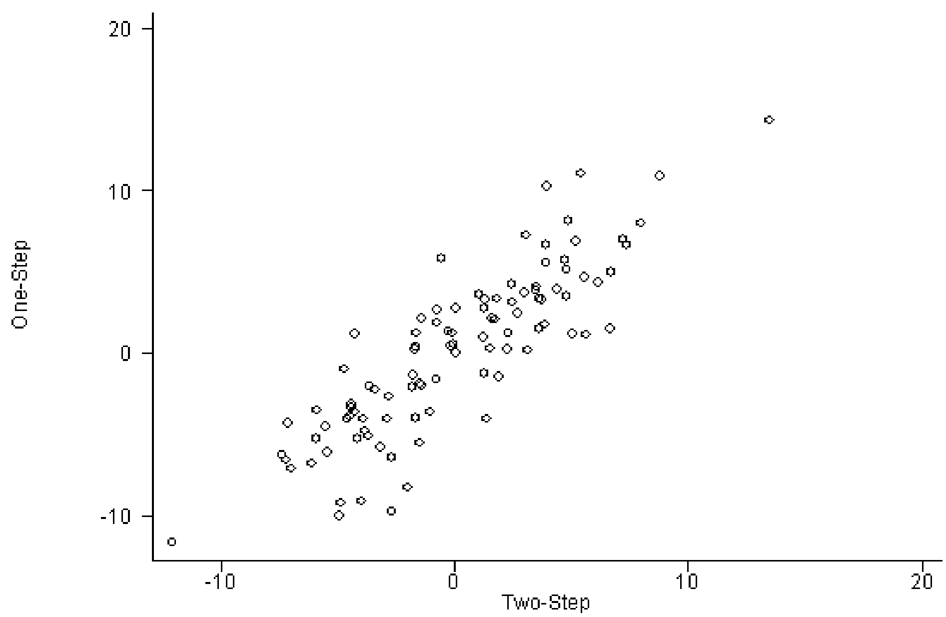
One-Step ECM Results

Variable	Estimate
(Constant)	20.18
	(4.56)
ΔX_{t-1}	0.07
	(0.12)
Y_{t-1}	-0.62
	(0.14)
X_{t-1}	0.27
	(0.12)

Note a couple things here:

- We find, unsurprisingly, the same results. The coefficient on Y_{t-1} is an estimate of ρ; here, it is nearly exactly the same as in the two-step model.
- The same is true of the predictions for the two models, which are very close to the same (see Figure 3).

Figure 3: Example: Predicted ΔYs, One- and Two-Step ECMs



In fact, asymptotically, the two methods are the same, and the same coefficient vectors can be recovered from each model. E.g., the two-step estimates:

$$\Delta Y_t = 20.18 + 0.07\Delta X_{t-1} - 0.62Y_{t-1} + 0.27X_{t-1} \tag{9}$$

are easily computed to yield:

$$\Delta Y_t = \hat{\beta}_0 + 0.07\Delta X_{t-1} - 0.62(\hat{\alpha} + Y_{t-1} - 0.44X_{t-1}) \tag{10}$$

which, not surprisingly, is very close to the estimates from the two-step model.[1]

The choice of one- or two-step methods isn't *usually* all that important. Beck (1993) argues in favor of the one-step approach, but largely on grounds of theory/causality (i.e., that one-step OLS is to be preferred when we have only one endogenous variable).

Next Time: Vector Autoregressions...

[1]Note that we can only recover the constant terms up to the linear function $\beta_0 + 0.62(\alpha) = 20.18$. Alternatively, if we omit the constant term from the cointegrating regression of Y on X, perhaps by "centering" the variables first, then $\hat{\beta}_0$ provides a direct estimate of β_0.